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The consistent result of cosmological constant from quantum cosmology and inflation with Born–Infeld scalar field

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Abstract. Quantum cosmology with a Born–Infeld (BI) type scalar field is considered. In the extreme limits of a small cosmological scale factor the wave function of the universe can also be obtained by applying the methods developed by Hartle–Hawking (HH) and Vilenkin. The HH wave function approach predicts that the most probable cosmological constant Λ equals $\frac{1}{\eta}$ ($\frac{1}{2\eta}$ equals the maximum of the kinetic energy of the scalar field). It is different from the original results ($\Lambda = 0$) for the cosmological constant obtained by Hartle–Hawking. The Vilenkin wave function predicts a nucleating universe with the largest possible cosmological constant, and it is larger than $1/\eta$. The conclusions can nicely be reconciled with cosmic inflation. We investigate the inflation model with the BI type scalar field and find that η depends on the amplitude of the tensor perturbation δ_h, having the form

$$\frac{1}{\eta} \simeq \frac{m^2}{12\pi\left[\left(\frac{9\delta_\Phi^2}{N\delta_h^2}\right)^2 - 1\right]}\,.$$

The vacuum energy in the inflation epoch depends on the tensor-to-scalar ratio δ_h/δ_Φ. The amplitude of the tensor perturbation δ_h may, in principle, be large enough to be discovered. However, it is only on the border of detectability in future experiments. If it will have been observed in the future, this will be very interesting as regards determining the vacuum energy in the inflation epoch.

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1 Introduction

Astronomical observations indicate that the cosmological constant is not zero and that it has the same order of magnitude as the matter energy density (the density parameter Ω_Λ of the vacuum energy ~ 0.73).

Before the year 1998 a crude experimental upper bound on Λ or the vacuum energy density ρ_V was provided: $\rho_V \lesssim 10^{-29}\,\mathrm{g/cm^3} \sim 10^{-47}\,\mathrm{GeV}^4$ [1]. However, when the universe approximates the Planck scale the energy difference between the symmetry and broken symmetry phase of the vacuum is 10^{18} GeV. The effective vacuum energy density $(10^{18}\,\mathrm{GeV})^4$ exceeds the observational limit by some 120 orders of magnitude. There are many symmetries that seem to be broken in the present universe, including chiral symmetry, electroweak symmetry and possibly supersymmetry. Each of these would give a contribution to ρ_V that would exceed the upper limit by at least forty orders of magnitude. It is very difficult to believe that the cosmological constant is fine tuned so that after all the symmetry breaking the effective vacuum energy density satisfies the upper bound. What one would like to have is some mechanism by which the cosmological constant Λ could relax to near zero. Weinberg has described five approaches to find such a mechanism, including anthropic considerations, superstrings and supersymmetry, adjustment mechanisms, changing gravity and quantum cosmology. At present, the approaches based on quantum cosmology are most promising [1]. In 1984 Hawking described how in quantum cosmology there could arise a distribution of values for the cosmological constant. Hawking introduces a 3-form gauge field $A_{\mu\nu\eta}$ or scalar field ϕ. According to the general ideas of Euclidean quantum cosmology, he obtained the result that the probability density is proportional to $\mathrm{e}^{3\pi/G\Lambda}$. The probability density has an infinite peak for $\Lambda \to 0_+$. The most probable cosmological constant will be one with a very small value [2]. Coleman considers the effect of topological fixtures known as wormholes. He argued that, if wormholes exist, they have the effect of making the cosmo-

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logical constant vanish [3, 4]. However, these conclusions have been hard to reconcile with cosmic inflation [5].

Recently, the problem of the cosmological constant based on quantum cosmology has been investigated by many authors. Kalinin and Melnikov discuss quantization of a closed isotropic cosmological model with a cosmological constant that is realized by the Wheeler–Dewitt (WDW) equation. It is shown that such a quantization leads to interesting results, in particular, to a finite lifetime of the system, and the appearance of the universe as a penetration via the barrier. These purely quantum effects appear when the cosmological constant is larger than zero [6]. Capozziello and Gavattini have defined the cosmological constant to be an eigenvalue of the WDW equation with $f(R)$ theories of gravity. The explicit calculation is performed for a Schwarzschild metric where the one-loop energy is derived by the zeta function regularization method and a renormalized running cosmological constant is obtained [7]. There are papers on quantum cosmology by Lemos and Monerat et al. [8, 9]. The quantization of the Friedmann–Robertson–Walker spacetime in the presence of a negative cosmological constant was used in the papers. The authors of these papers have concluded that there are solutions that avoid singularities (big bang, big crunch) at the quantum level [8, 9]. Moguigan has discussed a seesaw mechanism of the cosmological constant in the context of quantum cosmology [10]. FRW quantum cosmology in the non-Abelian Born–Infeld theory has been discussed by Moniz [11]. Reference [12] has discussed the quantum birth of the universe in a gravitational theory of varying cosmological constant.

In the 1930's Born and Infeld [38] attempted to eliminate the divergent self-energy of the electron by modifying Maxwell's theory. Born–Infeld electrodynamics follows from the Lagrangian

$$L_{\mathrm{BI}} = b^2\left(\sqrt{1-(1/2b^2)F_{\mu\nu}F^{\mu\nu}}-1\right),$$

where $F_{\mu\nu}$ is the electromagnetic field tensor. Our BI type Lagrangian has been first proposed by Heisenberg in order to describe the process of meson multiple production connected with the strong field regime [39–41]. The Born–Infeld type action also appears in string theory [42]. In the important paper [43] it was demonstrated that the leading-order term in the expansion in ∂F of the condition of conformal invariance of the open string sigma model follows indeed from the BI action. Static and spherically symmetric solutions of the BI type scalar field have been recently investigated qualitatively by Oliveiva [44]. Furthermore our BI type scalar field Lagrangian is the special case of the tachyon Lagrangian $u(\varphi)\sqrt{1-g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}}$ where the potential $u(\varphi) = \text{constant}$. Mukhanov and Vikman [34–36] have investigated inflation with an analogous scalar field Lagrangian

$$\alpha^2\left(\sqrt{1+\frac{g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}}{\alpha^2}}-1\right)-V(\varphi).$$

The speed of cosmological perturbations obeys $c_{\mathrm{s}}^2 > 1$, which is different with our BI scalar field Lagrangian (see (34)). We have investigated quantum cosmology and the dark energy model in our BI type scalar field [13–19].

In this paper, we combine inflation with quantum cosmology in the BI type scalar field and obtain the interesting result that the vacuum energy (and the corresponding cosmological constant) in the inflation epoch depends on the tensor-to-scalar ratio $\frac{\delta_h}{\delta_\Phi}$.

The present paper is organized as follows. In Sect. 2 we consider quantum cosmology with a BI type scalar field, obtain the Wheeler–Dewitt equation of our BI scalar field model. In Sect. 3 we apply Hartle–Hawking's method to obtain the wave function of the universe. Vilenkin's quantum tunneling approach is also considered. The probability density obtained from the Hartle–Hawking method is proportional to $\mathrm{e}^{3\pi/G\left(\Lambda-\frac{1}{\eta}\right)}$. The probability density has an infinite peak for $\Lambda \to \frac{1}{\eta}$ (the maximum kinetic energy $\frac{1}{2}\dot{\phi}^2$ is $\frac{1}{2\eta}$). i.e., the vacuum energy equals twice the maximum kinetic energy of the scalar field. The Vilenkin wave function predicts a nucleating universe with largest possible cosmological constant and it is larger than $1/\eta$. In Sect. 4 we discuss inflation with a BI type scalar field. We find that η depends on the amplitude of the tensor perturbation δ_h, with the form

$$\frac{1}{\eta} \simeq m^2/12\pi\left[\left(\frac{9\delta_\Phi^2}{N\delta_h^2}\right)^2-1\right].$$

We conclude our results in the last section.

2 WDW equation with BI type scalar field

The action of the gravitational field interacting with a Born–Infeld type scalar field is given by

$$S = \int \frac{R}{4\pi G}\sqrt{-g}\,\mathrm{d}^4x + \int L_{\mathrm{s}}\sqrt{-g}\,\mathrm{d}^4x\,, \tag{1}$$

where we have chosen units so that $c = 1$, R is the Ricci scalar curvature and the Lagrangian L_{s} of the BI scalar field [13–19] is

$$L_{\mathrm{s}} = \frac{1}{\eta}\left(1-\sqrt{1-\eta g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}}\right)-V(\varphi)\,, \tag{2}$$

where η is a constant and $V(\varphi)$ is the potential of the vacuum field. We shall assume that the potential of the vacuum field $V(\varphi)$ is a constant. Then from a physical point of view, it is equivalent to a cosmological constant.

At the Planck time, quantum effects played the main role in the universe. So it is appropriate to describe the dynamics and evolution of the very early universe by using the cosmological wave function $\Psi(h_{ij},\varphi)$ defined on the superspace of all three-metrics h_{ij} and matter fields φ. In superspace it satisfies the WDW equation:

$$\hat{H}\Psi = 0\,. \tag{3}$$

$\hat{H}$ is a second-order differential operator in superspace. In principle, $\Psi(h_{ij},\varphi)$ should contain the answer to all meaningful questions one can ask about the evolution of the very early universe. In order to find out the solution of the WDW equation, we shall apply the minisuperspace model – a Robertson–Walker (RW) space-time metric. The BI type scalar field is given by (2). In the minisuperspace there are only two degrees of freedom: $a(t)$ and $\varphi(t)$. The RW space-time metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1+kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right]. \tag{4}$$

Using (2) and by integrating with respect to the space components the action (1) becomes (the upper dot means the derivative with respect to the time t)

$$\begin{aligned} S &= \int \frac{3\pi}{4G}(1-\dot{a}^2)a\,\mathrm{d}t \\ &\quad + \int 2\pi^2 a^3\left[\frac{1}{\eta}(1-\sqrt{1-\eta\dot{\varphi}^2}) - V\right]\mathrm{d}t \\ &= \int \mathcal{L}_\mathrm{g}\,\mathrm{d}t + \int \mathcal{L}_\mathrm{s}\,\mathrm{d}t\,. \end{aligned} \tag{5}$$

From the Euler–Lagrange equation we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}_\mathrm{s}}{\partial\dot{\varphi}}\right) - \frac{\partial\mathcal{L}_\mathrm{s}}{\partial\varphi} = 0\,. \tag{6}$$

We can obtain

$$\dot{\varphi} = \frac{c}{\sqrt{a^6+\eta c^2}}\,, \tag{7}$$

where c is an integral constant. From the above equation we know that the cosmological scale factor $a(t)$ is very large or small when $\dot{\varphi}$ is very small or large, respectively. The maximum of the kinetic energy is $1/2\eta$ from (7).

To quantize the model, we first find the canonical momenta $P_a = \partial\mathcal{L}_g/\partial\dot{a} = -(3/2G)a\dot{a}$, $P_\varphi = \partial\mathcal{L}_\varphi/\partial\dot{\varphi} = 2\pi^2a^3\dot{\varphi}/\sqrt{1-\eta\dot{\varphi}^2}$ and the Hamiltonian $H = P_a\dot{a} + P_\varphi\dot{\varphi} - \mathcal{L}_g - \mathcal{L}_s$. H can be written as the follows:

$$\begin{aligned} H &= -\frac{G}{3\pi a}P_a^2 - \frac{3\pi}{4G}a\left[1-\frac{8\pi G}{3}a^2V(\varphi)\right] \\ &\quad - \frac{2\pi^2a^3}{\eta}\left[1-\sqrt{1+\frac{\eta P_\varphi^2}{4\pi^4a^6}}\right]. \end{aligned} \tag{8}$$

For $\dot{\varphi}^2 \ll \frac{1}{\eta}$, the Hamiltonian (8) can be simplified by using a Taylor expansion, and the terms smaller than $\dot{\varphi}^6$ can be ignored, so the Hamiltonian becomes

$$\begin{aligned} H &= -\frac{G}{3\pi a}P_a^2 - \frac{3\pi}{4G}a\left[1-\frac{8\pi G}{3}a^2V(\varphi)\right] + \frac{P_\varphi^2}{4\pi^2a^3} \\ &\quad - \frac{\eta P_\varphi^4}{64\pi^6a^9}\,. \end{aligned} \tag{9}$$

If $\dot{\varphi}$ is very large ($\dot{\varphi}^2 \sim 1/\eta$), (8) becomes

$$H = -\frac{G}{3\pi a}P_a^2 - \frac{3\pi}{4G}a\left\{1-\frac{8\pi G}{3}a^2\left[V(\varphi)-\frac{1}{\eta}\right]\right\}. \tag{10}$$

The WDW equation is obtained from $\hat{H}\psi = 0$, (9) and (10) by replacing $P_a \to -\mathrm{i}(\partial/\partial a)$ and $P_\varphi \to \mathrm{i}(\partial/\partial\varphi)$. Then we obtain

$$\left[\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial^2}{\partial\tilde{\Phi}^2} - \frac{\eta}{16\pi^4a^8}\frac{\partial^4}{\partial\tilde{\Phi}^4} - U(a,\tilde{\Phi})\right]\psi = 0 \tag{11}$$

and

$$\left[\frac{\partial^2}{\partial a^2} + \frac{p}{a}\frac{\partial}{\partial a} - u(a,\tilde{\Phi})\right]\psi = 0\,, \tag{12}$$

where $\tilde{\Phi}^2 = 4\pi G\varphi^2/3$ and the parameter p represent the ambiguity in the ordering of factor a and $\partial/\partial a$ in the first term of (9) and (10). The variation of p does not affect the solution of (9) and (10). In following discussion, we shall set $p = -1$ [20–22, 29–32]. We have also denoted

$$U(a,\tilde{\Phi}) = \left(\frac{3\pi}{2G}\right)^2 a^2\left[1-\frac{8\pi G}{3}a^2V(\tilde{\Phi})\right], \tag{13}$$

$$u(a,\tilde{\Phi}) = \left(\frac{3\pi}{2G}\right)^2 a^2\left\{1-\frac{8\pi G}{3}a^2\left[V(\tilde{\Phi})-\frac{1}{\eta}\right]\right\}. \tag{14}$$

Equations (11) and (12) are the WD equations corresponding to the action (1) in the case of a large and small scale factor a, respectively. If the scale factor a is very large, the general solution of (1) is given by

$$\psi(a,\varphi) \sim \frac{a}{a_0}Z_{\frac{1}{3}}\left(\frac{2\tilde{\mu}a^3}{3}\right)\mathrm{e}^{-k\varphi}\,, \tag{15}$$

where a_0 is the Planck length, k is an arbitrary constant, and $\tilde{\mu} = 2\pi^2V(\varphi)$. $\psi(a,\varphi)$ is an oscillatory function with respect to the scale factor a [13–19].

3 The Vilenkin and Hartle–Hawking method

Next we will use Vilenkin's quantum tunneling [20–22] approach to consider the cosmology in the case of very large $\dot{\varphi}$ (correspondingly very small $a(t)$). Equation (12) has the form of a one-dimensional Schrödinger equation for a "particle" described by a coordinate $a(t)$, which has zero energy and moves in a potential u. The classically allowed region is $u \le 0$ or $a \ge H^{-1}$, with

$$H = \left[\frac{8\pi G}{3}\left(V-\frac{1}{\eta}\right)\right]^{1/2}.$$

In this region, disregarding the pre-exponential factor, the WKB solutions of (12) are

$$\psi_\pm^{(1)}(a) = \exp\left\{\left[\pm\mathrm{i}\int_{H^{-1}}^{a} P(a')\,\mathrm{d}a'\right] \mp \frac{\mathrm{i}\pi}{4}\right\}. \tag{16}$$

The under-barrier ($a < H^{-1}$, classically forbidden or Euclidean region) solutions are

$$\psi_\pm^{(2)}(a) = \exp\left\{\left[\pm\int_{a}^{H^{-1}} |P(a')|\,\mathrm{d}a'\right]\right\}, \tag{17}$$

where $P(a) \equiv \sqrt{|-u(a)|}$.

The classical momentum conjugate to a is $P_a = -a\dot{a}$. For $a > H^{-1}$, we have

$$\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}a}\right)\psi_{\pm}^{(1)}(a) = \pm P(a)\psi_{\pm}^{(1)}(a)\,, \tag{18}$$

and thus $\psi_{-}^{(1)}(a)$ and $\psi_{+}^{(1)}(a)$ describe the expanding and contracting universe, respectively. The tunneling boundary condition requires that only the expanding component should be present at large a,

$$\psi_{\mathrm{T}}(a > H^{-1}) = \psi_{-}^{(1)}(a)\,. \tag{19}$$

The under-barrier wave function is found from the WKB connection formula

$$\psi_{\mathrm{T}}(a < H^{-1}) = \psi_{+}^{(2)}(a) - \frac{\mathrm{i}}{2}\psi_{-}^{(2)}(a)\,. \tag{20}$$

The growing exponential $\psi_{-}^{(2)}(a)$ and the decreasing exponential $\psi_{+}^{(2)}(a)$ have comparable amplitudes at the nucleation point $a = H^{-1}$, but away from that point the decreasing exponential dominates,

$$\psi_{\mathrm{T}}(a < H^{-1}) \approx \psi_{+}^{(2)}(a) = \exp\left[\frac{\pi}{2GH^2}(1 - H^2a^2)^{\frac{3}{2}}\right]. \tag{21}$$

The "tunneling amplitude" is proportional to

$$\frac{\psi_{\mathrm{T}}(H^{-1})}{\psi_{\mathrm{T}}(0)} = \mathrm{e}^{-\frac{\pi}{2GH^2}}\,. \tag{22}$$

The corresponding probability density obeys

$$P_{\mathrm{T}} \propto \mathrm{e}^{\frac{-\pi}{GH^2}}\,. \tag{23}$$

From (23) we obtain the result that the tunneling wave function predicts a nucleating universe with the largest possible vacuum energy (i.e., the largest possible cosmological constant), but the cosmological Λ must be larger than $1/\eta$. In other words, the vacuum energy is larger than twice the maximum of the kinetic energy of the scalar field. It is the correct condition for inflation. Equations (21) and (23) can be obtained by an alternative method, devised by Zeldovich and Starobinsky [23], Rubakov [24] and Linde [25–28]. Equation (23) predicts that a typical initial value of the field φ is given by $V(\varphi) \sim M_{\mathrm{P}}^4$ (if one does not speculate about the possibility that $V(\varphi) \gg M_{\mathrm{P}}^4$), which leads to a very long stage of inflation.

The Hartle–Hawking no boundary wave function is given by the path integral [29–32]

$$\psi_{\mathrm{HH}} = \int [\mathrm{d}g][\mathrm{d}\varphi]\mathrm{e}^{-S_{\mathrm{E}}(g,\varphi)}\,. \tag{24}$$

In order to determine ψ_{HH}, we assume that the dominant contribution to the path integral is given by the stationary points of the action (the instantons) and we evaluate ψ_{HH} simply as $\psi_{\mathrm{HH}} \sim \mathrm{e}^{-S_{\mathrm{E}}|_{\text{saddle-point}}}$. When $\dot{\varphi}$ is very large, $\dot{\varphi}^2 \sim 1/\eta$, from the action (5) we can obtain

$$S = \int \frac{3\pi}{4G}[(1 - \dot{a}^2)a]\,\mathrm{d}t - \int 2\pi^2 a^3\left(V - \frac{1}{\eta}\right)\mathrm{d}t\,. \tag{25}$$

The corresponding Euclidean action $S_{\mathrm{E}} = -\mathrm{i}(S)_{\text{continue}}$ is

$$S_{\mathrm{E}} = \int \frac{3\pi}{4G}\left[1 + \left(\frac{\mathrm{d}a}{\mathrm{d}\tau}\right)^2\right]a\,\mathrm{d}\tau + \frac{3\pi}{4G}\int a^3H^2\,\mathrm{d}\tau\,, \tag{26}$$

where $H^2 = \frac{8\pi G}{3}\left(V - \frac{1}{\eta}\right)$ and $\tau = \mathrm{i}t$. From action (25), we can also obtain that $a(t)$ satisfies the following classical equation of motion:

$$-\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 - 1 + H^2a^2 = 0\,. \tag{27}$$

The solution of (27) is the de Sitter space with $a(t) = H^{-1}\cosh(Ht)$. The corresponding Euclidean version (replacing $t \to -\mathrm{i}\tau$) of (27) is

$$\left(\frac{\mathrm{d}a}{\mathrm{d}\tau}\right)^2 - 1 + H^2a^2 = 0\,. \tag{28}$$

The solution of (28) is

$$a(\tau) = H^{-1}\sin(H\tau)\,. \tag{29}$$

We consider a saddle point approximation to the path integral (24), use (26) and (29), and obtain

$$\psi_{\mathrm{HH}}(a) \propto \exp\left[-\frac{\pi}{2GH^2}(1 - H^2a^2)^{\frac{3}{2}}\right]. \tag{30}$$

The only difference between the HH wave function (30) and Vilenkin's wave function (21) is the sign of the exponential factor. The HH wave function (30) gives the probability density

$$P_{\mathrm{HH}} \propto \mathrm{e}^{\frac{\pi}{GH^2}}\,. \tag{31}$$

The HH probability density (31) is the same as Vilenkin's one (23), except for a sign in the exponential factor. The probability density (31) is peaked at $V - (1/\eta) = 0$ (note that here $H^2 = \frac{8\pi G}{3}(V - 1/\eta)$) and it predicts a very possible universe with a positive cosmological constant $\frac{1}{\eta}$. The corresponding vacuum energy equals twice the maximum of the kinetic energy of the scalar field. It predicts the correct condition for inflationary cosmology. It is different from the previous result predicted by Hawking, that the cosmological constant equals zero [2].

4 Inflation with a BI type scalar field

The Lagrangian of the BI type scalar field is

$$L = K(X) - V = \frac{1}{\eta}\left[1 - \sqrt{1-2\eta X}\right] - \frac{1}{2}m^2\phi^2 . \quad (32)$$

We assume that for the homogeneous scalar field $X = \frac{1}{2}\dot{\phi}^2$. Here L plays the role of the pressure p. The corresponding energy density is

$$\rho = 2XL_{,X} - L = \frac{1}{\eta}\left[(1-2\eta X)^{-\frac{1}{2}} - 1\right] + \frac{1}{2}m^2\phi^2 . \quad (33)$$

The effective speed of sound (i.e., speed of propagation of the cosmological perturbations) is

$$c_s^2 = \frac{p_{,X}}{\rho_{,X}} = 1 - 2\eta X . \quad (34)$$

The stability condition with respect to high frequency cosmological perturbations requires $c_s^2 > 0$. We can find that $c_s^2 < 1$ and when $\eta \to 0$, $c_s^2 = 1$. In the nonlinear scalar field model considered by Mukhanov and Vikman [34–36], the speed of sound obeys $c_s^2 > 1$. Let us consider a RW spacetime with small perturbations:

$$\mathrm{d}s^2 = (1+2\Phi)\,\mathrm{d}t^2 - a(t)^2[(1-2\Phi)\delta_{ik} + h_{ik}]\,\mathrm{d}x^i\,\mathrm{d}x^k , \quad (35)$$

where Φ is the gravitational potential characterizing scalar metric perturbations and h_{ij} is a traceless, transverse perturbation describing gravitational waves. The equation of the Einstein gravitation field is

$$H^2 + \frac{k}{a^2} = \frac{8\pi}{3}\rho , \quad (36)$$

where $H^2 = (\frac{\dot{a}}{a})^2$, the dot denotes the derivative with respect to time t. In the inflation epoch, the term k/a^2 in (36) becomes negligibly small compared with H^2. The effective energy-moment conservation law is

$$\dot{\rho} + 3H(\rho + p) = 0 . \quad (37)$$

We obtain the following equation for the scalar field by varying the action of the BI type scalar field:

$$\ddot{\phi} + 3c_s^2 H\dot{\phi} + \frac{V_{,\phi}}{\rho_{,X}} = 0 . \quad (38)$$

From (36) and (38) for the field motion it is clear that if the slow-roll conditions

$$XK_{,X} \ll V , \quad K \ll V, |\ddot{\phi}| \ll \frac{V_{,\phi}}{\rho_{,X}} \quad (39)$$

are satisfied for at least 75 e-folds, then we have successful slow-roll inflation due to the potential $V(\varphi)$. Considering the canonical scalar field with $K = X$, one can take a flat potential $V(\phi)$ so that $X \ll V$ (for more than 75 e-folds). It is the standard slow-roll inflation [33] and in this case $c_s = 1$. In contrast to ordinary slow-roll inflation one can have any speed of sound and $c_s < 1$. It is important for the amplitude of the final scalar perturbations (during the postinflationary, radiation-dominated epoch) and the ratio of the tensor-to-scalar amplitudes on supercurvature scales is given by [34–36]

$$\delta_\Phi^2 \simeq \frac{64}{81}\left[\frac{\rho}{c_s(1+p/\rho)}\right]_{c_s k \simeq Ha} , \quad (40)$$

$$\frac{\delta_h^2}{\delta_\Phi^2} \simeq 27[c_s(1+p/\rho)]_{k\simeq Ha} . \quad (41)$$

Here it is worthwhile recalling that all physical quantities on the right hand side of (40) and (41) have to be calculated during inflation at the moment when perturbations with wave number k cross the corresponding horizon: $c_s k \simeq Ha$ for δ_Φ and $k \simeq Ha$ for δ_h, respectively. The amplitude of the scalar perturbations δ_Φ is a free parameter of the inflationary theory, which is taken to fit the observations (10^{-5}).

In the slow-roll regime, (36) and (38) reduce to

$$H \simeq \sqrt{\frac{4\pi}{3}}m\varphi , \quad (42)$$

$$3p_{,X}H\dot{\varphi} + m^2\varphi \simeq 0 . \quad (43)$$

From (42) and (43) we can obtain the slow-roll solution $\dot{\varphi} \simeq -\frac{mc_I}{\sqrt{12\pi}}$, and then we obtain

$$\varphi = \varphi_0 - \frac{mc_I}{\sqrt{12\pi}}t , \quad (44)$$

where $c_I = \left(1 + \frac{\eta m^2}{12\pi}\right)^{-1/2}$; it is the sound speed during inflation. The sound speed is smaller than the speed of light, approaching it as $\eta \to 0$. The effective energy density and pressure are given by

$$\rho = \frac{1}{\eta}\left[\frac{1}{c_I} - 1\right] + \frac{1}{2}m^2\varphi^2 , \quad p = \frac{1}{\eta}[1 - c_I] - \frac{1}{2}m^2\varphi^2 . \quad (45)$$

To determine $a(\varphi)$ we use $\dot{\varphi} \simeq -\frac{mc_I}{\sqrt{12\pi}}$ to rewrite (42) as

$$-\frac{mc_I}{\sqrt{12\pi}}\frac{\mathrm{d}\ln a}{\mathrm{d}\varphi} \simeq \sqrt{\frac{4\pi}{3}}m\varphi , \quad (46)$$

and obtain

$$a(\varphi) \simeq a_f \exp\left[\frac{2\pi}{c_I}\left(\varphi_f^2 - \varphi^2\right)\right] , \quad (47)$$

where a_f and φ_f are the values of the scale factor and the scalar field at the end of inflation. The inflation is over when $(\rho+p)/\rho \simeq c_I/(6\pi)^{1/2}$ becomes of order unity, that is, at $\varphi \sim \varphi_f = (c_I/6\pi)^{1/2}$. After that the field φ begins to oscillate and decays. Given a number of e-folds before the end of inflation N, we find that at this time

Table 1. Twelve selected values of $\frac{\delta_h}{\delta_\phi}$ with different value of $\frac{1}{\eta}$

$\frac{1}{\eta}$	1	10^{-3}	10^{-5}	10^{-7}	10^{-9}	10^{-11}
$\frac{\delta_h}{\delta_\Phi}$	0.3464101614	0.3464101614	0.3464101614	0.3464101602	0.3464100291	0.3463969309
$\frac{1}{\eta}$	10^{-13}	10^{-15}	10^{-17}	10^{-19}	10^{-21}	10^{-23}
$\frac{\delta_h}{\delta_\Phi}$	0.3450994646	0.2747269024	0.09836927818	0.03115736170	0.009852982485	0.003115787144

$2\pi\varphi^2/c_\mathrm{I} \sim N$; hence $(\rho+p)/\rho \simeq 1/3N$ does not depend on c_I. Thus, for a given scale, which crosses the Hubble scale N e-folds before the end of inflation, the tensor-to-scalar ratio is [34–36]

$$\frac{\delta_h^2}{\delta_\Phi^2} \simeq 27[c_\mathrm{I}(1+p/\rho)] \simeq \frac{9c_\mathrm{I}}{N}\,. \tag{48}$$

Next one can estimate the mass m that is necessary to produce the observed $\delta_\Phi \sim 10^{-5}$. Using $(2\pi\varphi^2/c_\mathrm{I}) \simeq N$, $(\rho+p)/\rho \simeq 1/3N$ and from (40) one can obtain $m \simeq 3\sqrt{3\pi}\delta_\Phi/4N$. Then one can obtain $m \sim 2.4\times 10^{-7}$ for $N \sim 75$. It is similar to the usual chaotic inflation [37]. The spectral index of the scalar perturbations is

$$n_\mathrm{s} - 1 \simeq -3\left(1+\frac{p}{\rho}\right) - H^{-1}\frac{\mathrm{d}\left[\ln\left(1+\frac{p}{\rho}\right)\right]}{\mathrm{d}t} = -\frac{2}{N}\,. \tag{49}$$

This is exactly the same tilt as for the usual chaotic inflation. Finally we obtain by (48) and $c_\mathrm{I} = \left(1+\frac{\eta m^2}{12\pi}\right)^{-\frac{1}{2}}$

$$\frac{1}{\eta} \simeq \frac{m^2}{12\pi\left[\left(\frac{9\delta_\Phi^2}{N\delta_h^2}\right)^2 - 1\right]}\,. \tag{50}$$

We list the details in Table 1.

From our Vilenkin and Hartle–Hawking wave function ((21) and (30)) we obtain $\Lambda \geq \frac{1}{\eta}$. In the inflation epoch, the evolution of the field φ is very slow, so that this field acts only as a cosmological constant $\Lambda(\varphi) = \frac{8\pi V(\varphi)}{M_\mathrm{P}^2}$, where M_P is the Planck mass [37]. Note that $\Lambda \geq \frac{1}{\eta} = 1, 10^{-3}, \cdots, 10^{-23}$ correspond to the vacuum energy $\rho_\mathrm{V} \geq \frac{M_\mathrm{P}^4}{8\pi}, \frac{10^{-3}M_\mathrm{P}^4}{8\pi}, \cdots, \frac{10^{-23}M_\mathrm{P}^4}{8\pi}$. From Fig. 1 or Table 1 we can conclude that only when the vacuum energy $\rho_\mathrm{V} < \frac{10^{-14}M_\mathrm{P}^4}{8\pi}$, the value of $\frac{\delta_h}{\delta_\Phi}$ begins to decrease rapidly. When $\rho_\mathrm{V} \to 0$, $\frac{\delta_h}{\delta_\Phi} \to 0$. The tensor perturbations can be seen indirectly in the B-mode of the CMB polarization. The amplitude of the tensor perturbations can, in principle, be large enough to be observed. However, it is only on the border of detectability in future experiments. If it will have been observed in future, this is very interesting as regards the definition of the cosmological constant.

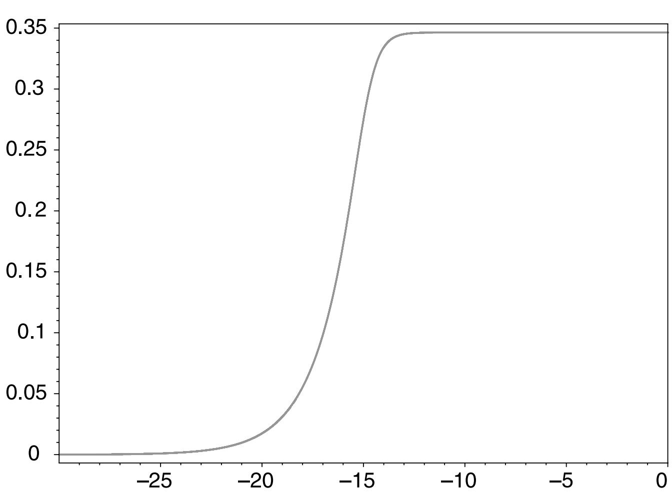

Fig. 1. The value of $\frac{\delta_h}{\delta_\Phi}$ with $\frac{1}{\eta}$. The horizontal axis represents $\log_{10}\frac{1}{\eta}$ and the vertical axis represents $\frac{\delta_h}{\delta_\Phi}$

5 Conclusion

Weinberg has described five directions that have been taken in trying to solve the problem of the cosmological constant. The five approaches respectively are anthropic considerations, superstrings and supersymmetry, adjustment mechanisms, changing gravity and quantum cosmology. At present, all of the five approaches to the cosmological constant problem remain interesting. The approach that is based on quantum cosmology is most promising [1]. In quantum cosmology with a BI type scalar field, our Hartle–Hawking wave function predicts that the most probable value of the cosmological constant is $\frac{1}{\eta}$, while our Vilenkin wave function predicts $\Lambda > \frac{1}{\eta}$. These results are the correct condition for inflationary cosmology. The parameter η can be obtained by a tensor perturbation from our inflation model. The tensor perturbations can be seen indirectly in the B-mode of the CMB polarization. The amplitude of the tensor perturbations can, in principle, be large enough to be observed. However, it is only on the border of detectability in future experiments. If it will have been observed in the future, this is very interesting as regards the determination of the cosmological constant in the inflation epoch.

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